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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

III. THE CREATION OF A THEORY OF LOGARITHMS OF COMPLEX NUMBERS BY EULER.

1747–1749.

It is desirable to look back, for a moment, over the 35 years of history of logarithms of negative and complex numbers. Thus far only three mathematicians have attempted to unravel the mysteries of this subject, namely Leibniz, John Bernoulli I and Euler. Their discussions were carried on entirely by letter; these letters were not published at the time. No articles or memoirs on this controversy had reached the press. The question had not been brought to the attention of the mathematical public.

Publicity on this subject began in 1745. In that year was published the correspondence between Leibniz and John Bernoulli I.[1] The reading of that correspondence acted as a tremendous stimulus upon Euler. As a boy of 20 he himself, as we have seen, had corresponded on this subject with his revered master, John Bernoulli I. That correspondence had set bare serious difficulties of the subject, but had not removed them. Since that time he had discovered the exponential expressions for sin x, cos x, and cos $x + i$ sin x; he had acquired a deeper insight into the properties of imaginary numbers. It was in 1745 that he completed his manuscript on the *Introductio,* which was issued from the press three years later. Two years later (in 1747) he carried on his classic researches on logarithmic theory. He discussed this subject in his correspondence with D'Alembert. Some of Euler's letters on this subject have been published only recently. For that reason not even Lampe's historical essay is based on a study of all the material that is now available. For convenience of reference we give Euler's writings on this subject, the year in which they were penned, and the year of publication. The first two publications we have already reviewed.

[1] *Virorum celeberr. Got. Gul. Leibnitii et Johan. Bernoullii commercium philosophicum et mathematicum. Lausannæ et Genevæ,* 1745.

	Time of Writing.	Time of Publication.
1. Correspondence between Euler and John Bernoulli I	1727–29	1902
2. Euler's *Introductio in analysin*	1745	1748
3. Euler's letters to D'Alembert	1747–8	1886[1] 1907[2]
4. Euler's article *Sur les logarithmes*	1747	1862[3]
5. Euler's letters to D'Alembert, dated Dec. 27, 1748; Jan. 3, 1750; July 26, 1763; Dec. 20, 1763		1768[4]
6. Euler's letter to D'Alembert .... Feb. 15,	1748	1911[5]
7. Euler's article *De la controverse entre* MRS. LEIBNITZ ET BERNOULLI	1749	1751[6]
8. Euler's article *Recherches sur les racines imaginaires*	1749	1751[7]

Euler's letter of April 15, 1747, to D'Alembert, written in French, is seen from the context to be in response to an argument advanced by D'Alembert in letters now lost. The correspondence between the two men seems at times to have been quite lively. We give the synopsis of five letters written by Euler from Berlin.

April 15, 1747. Euler to D'Alembert: I must oppose your argument that e^1 can have a positive and a negative value. I admit the value to be arbitrary, say 10 or 2.718 . . . , but as soon as e in $y = e^x$ is assigned a definite value, the system of logarithms of all numbers is fixed, as is also the curve $y = e^x$. One cannot assign e two different values at the same time, without making the resulting curve the composite of two distinct curves. If $e = 1 + 1 + 1/(1\cdot 2) + 1/(1\cdot 2\cdot 3) + \cdots$ then it is clear that the logarithms of negative numbers must be "impossible," for no value of x can be found which makes e^x or $1 + x/1 + xx/1\cdot 2 + \cdots$ negative. To you it seems paradoxical that $ly = l(-y)$, but for *any* value of a we have $d(ly) = d(lay)$, hence also for $a = -1$. The argument by which you prove that $l(-1) = 0$ enables you to prove that $l\sqrt{-1} = 0$; for since $\sqrt{-1}\cdot\sqrt{-1} = -1$, you have $l\sqrt{-1} + l\sqrt{-1} = l(-1) = 2l\sqrt{-1} = \frac{1}{2}\log(+1)$ and $l\sqrt{-1} = \frac{1}{4}l(+1) = 0$. You will admit that the logarithms of imaginary numbers are not real; else $(l\sqrt{-1})/\sqrt{-1}$ could not express the quadrature of the circle. Let $(l\sqrt{-1})/\sqrt{-1} = \alpha$, then $l\sqrt{-1} = \alpha\sqrt{-1}$, which is imaginary. Now, if $l\sqrt{-1}$ is imaginary, why should not $2l\sqrt{-1} = l(-1)$ be so? Similarly it would follow that the logarithm of an imaginary cube root of 1 would be 0, as are also $l(+1)$, $l(-1)$, $l\sqrt{-1}$, etc. This is untenable. You will reply that even $l(+1)$ must be imaginary, for $l(+1) = 2l(-1) = 4l\sqrt{-1} = 3l\,(-1+\sqrt{-3})/2 = \cdots$. This is exactly what I claim, for $l(+1)$ has an infinite number of distinct values, of which all are imaginary except 0. Let $l(+1) = 0, \alpha, \beta, \gamma, \delta, \epsilon, \cdots$, then $\frac{1}{2}\alpha, \frac{1}{2}\gamma, \frac{1}{2}\epsilon, \cdots$ are the logarithms of -1, all imaginary; not every half of the values of $l(+1)$ is a value of $l(-1)$; for, -1 is only one value of $\sqrt{+1}$. The other value $+1$ has the logarithms $\frac{1}{2}\cdot 0, \frac{1}{2}\beta, \frac{1}{2}\delta, \cdots$, which are same as $0, \alpha, \beta, \gamma, \delta, \epsilon, \cdots$; for we have $\frac{1}{2}\beta = \alpha$, $\frac{1}{2}\delta = \beta, \cdots$. Similarly for the logarithms of the cube roots of 1. I can give the actual values. Letting π be the circumference (!) of a circle of unit radius, the values of $\log(+1)$ are: $0, \pm\pi\sqrt{-1}, \pm 2\pi\sqrt{-1}, \pm 3\pi\sqrt{-1}, \cdots$. Those of $l(-1)$ are: $\pm\frac{1}{2}\pi\sqrt{-1}, \pm\frac{3}{2}\pi\sqrt{-1}, \pm\frac{5}{2}\pi\sqrt{-1}, \cdots$. Generally: $l(1^p) = \pi(mp+n)\sqrt{-1}$, $l(-1)^p = \pi(\frac{1}{2}p + mp + n)\sqrt{-1}$, where m and n are $+$ or $-$ integers. Here all difficulties disappear which arise in the effort to make the logarithms of negative numbers real. According to the formula $2l(-1) = l(+1) = 0$ we would have $l\sqrt{-1} = 0$ and $l\frac{1}{2}(-1+\sqrt{-3}) = 0$. You say that in $e^x = y$, y may be

[1] *Bullettino* BONCOMPAGNI, Vol. 19, 1886, pp. 136–148.

[2] E. LAMPE in *Festschr. z. Feier d. 200. Geburtstages* L. EULERS, Leipzig und Berlin, 1907, pp. 125–135.

[3] EULER, *Opera posthuma*, 1 (1862), pp. 269–281.

[4] *D'Alembert, Opuscules mathématiques*, Vol. IV, Paris, 1768, pp. 342–343, 146, 162.

[5] *Bibliotheca mathematica*, 3d S., Vol. 11, 1911, pp. 220–226.

[6] *Mém. de l'acad. de sc. de Berlin*, Vol. 5 (1749), 1751, pp. 139–179.

[7] Same volume, pp. 270–288.

$+$ and also $-$, when $x = \frac{1}{2}$. But since e^x represents the value of the series $1 + x + x^2/(1 \cdot 2) + \cdots$, I rest on solid ground when I say that e^x stands for only one value, which is positive even when x is fractional.

The logarithms of negative and imaginary numbers had been under discussion, off and on, by a few mathematicians for over 35 years, but the above letter of Euler to D'Alembert is the first investigation which really penetrates the subject and yields substantial results. The theorem was announced therein that Log n has an infinite number of logarithms which are all imaginary, except when n is a positive number, in which case one logarithm out of this infinite number is real. It is also interesting to observe that Euler gives in this letter a new definition of the exponential e^x. He lets e^x represent the value of the exponential series, $1 + x + x^2/(1 \cdot 2) + \cdots$, and thereby he takes a point of view which has since found adoption in the theory of functions. D'Alembert's reply to Euler's letter is not known. Since Euler's main results were stated in the above letter without proof, it is not surprising if D'Alembert was not convinced. Some idea of D'Alembert's position may be gathered from Euler's next letter.

Aug. 19, 1747. Euler to D'Alembert: In your article on integrals, recently published in the second volume of our memoirs, I have crossed out the paragraph on log (-1), as you directed. Your statement, that $l(-x)$ can be expanded into a series whose value is real, is incomprehensible to me. When you say that e must be considered, not a parameter of the logarithmic curve, but the ordinate when $x = 1$, that, therefore, e may be $+$ as well as $-$, then I have as much right to claim that the logarithmic curve has not only two branches, but any number. If $x = l(+y)$ and $x = l(-y)$ yield two branches, then $x = l(my)$, the differential equation of which is the same for all values of m, yields a branch for every possible value of m. In e^x you take $x = k/g$, where $k : g =$ odd number : even number. With equal right we can take $k : g =$ even number : odd number, or $=$ odd number : odd number, and thereby reach conclusions at variance with yours. Your arguments fail to establish the formula $l(+x) = l(-x)$. As regards $(l\sqrt{-1})/\sqrt{-1}$, I maintain that it can have no other values than $\frac{1}{2}(4n+1)\pi$, where n is any integer, π the circumference of a circle of unit diameter; hence this formula can never yield 0. I have sent to the Academy a memoir, which appears to me to remove all difficulties on this subject which formerly perplexed me greatly.

P. S. You admit that $l(+1) = \pm 2n\pi\sqrt{-1}$ and $l(-1) = \pm (2n-1)\pi\sqrt{-1}$, but you claim that 0 is among the logarithms of -1. Since twice the logarithm of -1 is $l(+1)$, we would have $l(+1) = \pm 2n\pi\sqrt{-1}$ and also $= \pm(2n-1)\pi\sqrt{-1}$. You admit that $l\sqrt{-1} = \pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$, and $l(-\sqrt{-1}) = \pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$, but assert that these formulæ do not yield all the logarithms, that they omit the logarithm 0. If so, then $l(+1)$, or $l(+\sqrt{-1}) + l(-\sqrt{-1})$, may equal $\pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$. You claim that 0 may be the logarithm of the higher imaginary roots of unity; hence $l(+1)$ would be represented by $a\sqrt{-1}$, whatever number a may be. Consequently, $l(+1)$ would be wholly indeterminate. This conclusion overthrows your contentions. My results are in perfect harmony with each other and yield no such indeterminateness.

Dec. 30, 1747. Euler to D'Alembert: I learn through Mr. de Maupertius that you are suspending mathematical research, in order to recuperate from ill-health. I shall therefore not trouble you with matters about imaginary logarithms. I have hardly anything to add to what I said before. I doubt whether my paper on this subject will remove all the doubts which you have brought.

Feb. 15, 1748. Euler to D'Alembert: The equation $y = 2^x$ gives a continuous curve above the axis of x, but if $x = \frac{1}{2}$, then $y = +\sqrt{2}$ and $-\sqrt{2}$, and I grant that there is a conjugate point below the axis of x. Taking $x = n/2$, there is an infinity of such points below the x-axis, which are isolated from each other.[1] The equation $y = (-2)^x$ gives an infinity of

[1] " . . . je pretend que chacun de ces points est isolé sans liaisons avec les voisins, quoique leurs distances soient meme infiniment petites."

isolated points and no continuous curve whatever. $y = e^x$ represents a continuous curve above the x-axis and isolated points below.

It is seen that Euler recedes here from his definition of e^x as the sum of the series $1 + x + x^2/(1 \cdot 2) + \cdots$, adopted in his letter of April 15, 1747. He assigns no reason for his change of view-point, nor does he explain how this admission affects the question under controversy.

Sept. 28, 1748. Euler to D'Alembert: The subject of imaginary logarithms is no longer so clearly in my mind that I could reply rigorously to your recent remarks. I must wait till I can examine this matter again.

The article which Euler reports in his letter of August 19, 1747, as having been sent by him to the Berlin Academy, is without doubt the article [1] which was first published in 1862 and entitled *Sur les logarithmes des nombres négatifs et imaginaires*.[2] Euler begins by referring to the "grande controverse" between Leibniz and John Bernoulli I which ended in disagreement between these great men who were in perfect harmony on all other parts of analysis. The glory of infallibility of this science receives a severe blow, if it presents questions in which it is impossible to ascertain the truth and end all dispute. It is my hope by the present research to settle the controversy on the logarithms of negative numbers. Reviewing the arguments of Leibniz and John Bernoulli I, Euler remarks that Leibniz's demur to B.'s argument—that $l(+ x) = l(- x)$ since $dlx = dl(- x) = dx/x$, on the ground that the rule for finding lx holds only when x is positive—shakes the very foundation of analysis, whose rules of operation are taken to be general and applicable to quantities of all kinds. B.'s error lies in claiming $lx = l(- x)$ on the ground that $dlx = dl(- x)$; the same argument yields the absurdity $lnx = lx$. B.'s claim, that the logarithmic curve is double, involves the argument just mentioned; if $lx = l(- x)$ then similarly $lnx = lx$, and the curve has an infinite number of branches. A differential equation always yields on integration a family of curves. B. and others claim that the asymptote to the logarithmic curve is a diameter of it, saying that $dx = dy/y^n$ yields a curve which in general has a diameter when n is odd and has one therefore when $n = 1$. Euler explains this point and closes with the novel statement that the question of the doubleness of the logarithmic curve is not necessarily connected with the question whether $l(- x)$ is real or imaginary. Euler then defines a logarithm; if $x = ly$, then $y = e^x$, e being a constant. "Donc le logarithme x d'un nombre proposé $= y$ n'est autre chose que l'exposant de la puissance de e qui est égale au nombre y" (p. 273). In hyperbolic logarithms, if ω is infinitesimally small, then $l(1 + \omega) = \omega$ and $e = 2{,}718 \cdots$. If y is negative, no real value of x satisfies $y = e^x$. To be sure, when $x = \frac{1}{2}$, $y = \pm \sqrt{e}$; but when $x = 2$, it is not true that $y = \pm\, ee$ and that $x = l \pm ee$. Hence, the statement that the logarithms of negative numbers are real, is certainly not a general truth. "Mais pour ce qui regarde l'ambiguité de la formule e^x, dans le cas oú x est une fraction d'un dénominateur pair, je ne sais pas si on la peut admettre dans les logarithmes.

[1] See G. Eneström, "Verzeichniss d. Schriften L. Eulers," Leipzig, 1910, *Jahresber. d. deutschen Math. Vereinig., Ergänzsb., IV Bd., 1 Lief.*, pp. 42, 202, Nos. 168, 807.

[2] L. Euleri *Opera posthuma*, Tomus prior, Petropoli, 1862, pp. 269–281.

Car, ayant égard à la nature et à l'usage des logarithmes, il semble qu'à chaque logarithme ne puisse répondre qu'un seul nombre" (p. 274). If some one says that $e^0 = e^{0/2} = \sqrt{e^0} = \sqrt{1} = \pm 1$, then by the same argument $x^1 = x^{2/2} = \pm x$, "et de plus que $a + x$ serait la même chose que $a - x$," whence the absurdity "toutes les quantités sont égales entre elles." But if $l(-1)$ is not 0, it must be imaginary, and $l(-y) = l(-1) + ly$ must be imaginary. That x can be the logarithm of only one number y is confirmed by the series $e^x = 1 + x + x^2/(1\cdot 2) + \cdots$. If one insists that, for $x = \frac{1}{2}$, $e^x = \pm\sqrt{e}$ it follows that the logarithms of imaginary numbers are real; for, when $x = \frac{1}{3}$, $\frac{1}{3}$ is the logarithm of the real cube root of e, as well as of the two imaginary ones. But B. has made the beautiful discovery that $\pi = 2(l\sqrt{-1})/\sqrt{-1}$; it follows that $l\sqrt{-1}$ must be imaginary and that $l\sqrt{-1} = 0$ cannot be true. But if we let $l(-1) = p$, p imaginary, we encounter $l(-1)^2 = l(+1) = 2p = 0$, which is contrary to hypothesis and leads up to $l\sqrt{-1} = 0$, as before. "Voilà donc des contradictions assez palpables qu'on recontre, de quelque côté qu'on se tourne; . . . ce serait sans doute une tache indélébile dans l'analyse, si la doctrine des logarithmes était tellement remplie de contradictions, qu'il fût impossible de trouver une conciliation" (p. 275). Euler tries the experiment of letting the logarithms of the three cube roots of y be different from each other, namely equal to $x/3$, $x'/3$ and $x''/3$. Of these three logarithms the first is real, the other two imaginary; but the triple of each must be x. "Cette explication me paraissait bien extrêmement paradoxe et insoutenable, mais pourtant moins absurde que les contradictions que j'aurais été obligé d'admettre dans la théorie des logarithmes des nombres négatifs et imaginaires" (p. 276). But the fertile mind of Euler had not exhausted all the possibilities; he comes out with the pregnant remark: "Après avoir bien pesé toutes les difficultés que je viens d'étaler, j'ai trouvé qu'elles ne viennent que de ce que nous supposons que chaque nombre n'a qu'un seul logarithme" (p. 276). Admitting that a number may have many, in fact an infinite number, of logarithms, all difficulties vanish. To show that this number is really infinite, Euler takes the circle as better suited for the study of logarithms than the logarithmic curve; he develops by the aid of the integral calculus the equation $\sqrt{-1}\varphi = l(\cos\varphi + \sqrt{-1}\sin\varphi)$, where he writes for φ the more general value $\varphi \pm 2n\pi$, n being any integer. From this formula he derives the logarithms of 1, -1, $-a$, $\sqrt{-1}$, and remarks that Leibniz was correct in claiming that the logarithms of negative numbers are imaginary. He tests his results on products, quotients and powers of positive, negative and imaginary numbers, and remarks, "l'on trouvera constamment un merveilleux accord avec la vérité" (p. 280). Using De Moivre's theorem he obtains finally

$$l1^{\mu/\nu} = \frac{1}{\nu}(\pm 2\mu m \pm 2\nu n)\pi\sqrt{-1}$$

and

$$l(-1)^{\mu/\nu} = \frac{1}{\nu}(\mu \pm 2\mu m \pm 2\nu n)\pi\sqrt{-1},$$

where m and n are any integers.

In following the history of logarithms after 1747, the reader must not forget that the preceding article of Euler was not published until 1862. Had it been published at the time when it was written, it doubtless would have contributed to a speedier settlement of the great controversy. As we shall see, Euler rewrote this article and prepared a longer and more exhaustive, though in some respects less convincing article, which he had printed in the Berlin memoirs for the year 1749. The article of 1747 has not received the attention it deserves. Cantor does not appear to have been aware of its existence.[1] Lampe[2] is of the opinion that the 1747 article is the one published in 1749. Felix Müller[3] states that the publication of 1862 is a continuation of the article of 1749, a statement seen at a glance to be untrue by any one who has read both articles. The interesting question arises, why was the first article not published soon after it was sent to the Berlin Academy, instead of being cast aside for posthumous publication? We can only guess at the reasons. Euler's aim was, as he says, to bring order and harmony into a subject which had been full of contradictions. Our guess is that Euler became dissatisfied with his article. Possibly the use of the integral calculus in the establishment of $i\varphi = \log(\cos\varphi + i\sin\varphi)$ appeared inappropriate for what seemed to be an elementary topic. In the article itself he expressed a doubt as to how the doubleness of the logarithmic curve affected logarithmic theory. The article does not explain fully how $2l(-a)$ is equal to la^2, and $2l(+a)$ is equal to la^2, and yet $2l(-a)$ is *not* equal to $2l(+a)$. In the paper of 1749 this matter is explained fully. If we examine Euler's letters to D'Alembert, we find that on Aug. 19, 1747, Euler wrote that he eliminated from an article of D'Alembert sent for publication in the Berlin memoirs a paragraph on the logarithms of negative numbers. This was done at D'Alembert's request. Yet what would be more natural than for Euler to withhold from print his own paper, if he thought it incomplete? On December 30, 1747, Euler expresses to D'Alembert his fear that his own paper would not remove all of D'Alemberts' doubts. On September 28, 1748, Euler makes the astonishing statement that he is not able to reply rigorously to some of D'Alembert's recent remarks and that he must wait until he can re-examine the subject.

Unfortunately Euler's second paper does not contain all the good things found in the first paper. In the first, Euler takes the reader fully into his confidence and gives a heart to heart talk. He lets the reader see how he struggles with the subject, how he carries on his experimentation. When he showed that $\log(-1)$ cannot be a real value, nor even an imaginary value, it looked as if all possibilities were exhausted. But he persists and tries the experiment of assuming that two of the cube roots of a positive number have each an imaginary logarithm, while the third root has a real logarithm. This leads to results which he considers a little less absurd than some previous results, but nevertheless *absurd*. Finally it dawned in Euler's mind that possibly the error lay in the

[1] Cantor, *op. cit.*, Vol. III, 2. aufl., 1901, p. 722.

[2] *Festschr. z. Feier d. 200. Geburtst. L. Eulers*, 1907, pp. 120, 131.

[3] Same *Festschrift*, p. 97.

assumption that a number had only one logarithm. He tried the assumption that there was an infinite number of them, and this worked! How easy all this is to a modern reader, but how hard it was to Euler. His labors in connection with this great research remind us of Kepler's struggle with the orbit of Mars. There is a difference, to be sure. Kepler had to construct a theory which would conform with the facts of nature as revealed by Tycho Brahe's telescope; Euler had to construct a theory which would conform with the demands for consistency in logic. Aside from this the procedure was the same. Kepler made several successive guesses as to what the orbit of Mars might be, only to find that he guessed wrong. Finally it occurred to him to try an *ellipse;* at last he found that he had guessed correctly. The reader of Euler's works will find that in other mathematical topics as well, particularly in the theory of numbers, Euler often followed methods closely akin to those of the student of natural science. And yet, Sir William Hamilton and Thomas Huxley would have us believe that mathematics is a science which knows nothing of observation and experiment!

Euler's paper of 1749 bears the title, *De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires.*[1] It starts out with a critical historical account of the controversy. The strictly constructive part of the article consists of a theorem and four problems. The theorem is that *there is an infinity of logarithms for every number.* The proof of this theorem was based in the 1747 article upon the relation $i\varphi = \log(\cos\varphi + i\sin\varphi)$. Here in the 1749 article it is based upon the assumption that $l(1+\omega) = \omega$, ω being infinitely small. Presumably this relation was based on the series $\log(1+x) = x - x^2/2 + x^3/3 - \cdots$. Euler's proof of 1749 plays so large a part in logarithmic theory during the half century following, that it should be reproduced here in full. As will be seen later, Euler's reasoning failed to carry conviction. The proof is as follows:

> Je me bornerai ici aux logarithmes hyperboliques, puisqu'on sait que les logarithmes de toutes les autres espèces sont à ceux-cy dans un rapport constant, ainsi quand le logarithme hyperbolique du nombre x est nommé $= y$, le logarithme tabulaire de ce même nombre sera $= 0,4342944819\cdot y$. Or le fondement des logarithmes hyperboliques est, que si ω signifie un nombre infiniment petit, le logarithme du nombre $1+\omega$ sera $= \omega$, ou que $l(1+\omega) = \omega$. De là il s'ensuit que $l(1+\omega)^2 = 2\omega$; $l(1+\omega)^3 = 3\omega$, & en général $l(1+\omega)^n = n\omega$. Mais puisque ω est un nombre infiniment petit, il est évident, que le nombre $(1+\omega)^n$ ne sauroit devenir égal à quelque nombre proposé x, à moins que l'exposant n ne soit un nombre infini. Soit donc n un nombre infiniment grand, & qu'on pose $x = (1+\omega)^n$, & le logarithme de x, qui a été nommé $= y$, sera $y = n\omega$. Donc pour exprimer y par x, la première formule donnant $1+\omega = x^{1/n}$ & $\omega = x^{1/n} - 1$, cette valeur étant substituée pour ω dans l'autre formule produira $y = nx^{1/n} - n = lx$. D'où il est clair que la valeur de la formule $nx^{1/n} - n$ approchera d'autant plus du logarithme de x, plus le nombre n sera pris grand; & si l'on met pour n un nombre infini, cette formule donnera la vraye valeur du logarithme de x. Or comme il est certain, que $x^{\frac{1}{2}}$ a deux valeurs différentes, $x^{\frac{1}{3}}$ trois, $x^{\frac{1}{4}}$ quatre, & ainsi de suite, il sera également certain, que $x^{1/n}$ doit avoir une infinité de valeurs différentes, puisque n est un nombre infini. Par conséquent cette infinité de valeurs différentes de $x^{1/n}$ produira aussi une infinité de valeurs différentes pour lx, de sorte que le nombre x doit avoir une infinité de logarithmes. C. Q. F. D.

Letting in $y = nx^{1/n} - n = lx$, $x = 1$, he gets $(1 + y/n)^n - 1 = 0$. The factors of any binomial $p^n - q^n = 0$ can be gotten from

$$p = q(\cos 2\lambda\pi/n \pm \sqrt{-1}\sin 2\lambda\pi/n), \text{ where } \lambda = \pm 1, \pm 2, \cdots.$$

[1] *Histoire d. l'acad. d. sc. et belles let.*, année 1749, Berlin, 1751, pp. 139–179.

Hence

$1 + y/n = \cos 2\lambda\pi/n \pm \sqrt{-1} \sin 2\lambda\pi/n$, and $y = \pm 2\lambda\pi\sqrt{-1}$, when $n = \infty$.

Euler proceeds to find the formulæ for $\log (\pm a)$, $\log (a + b\sqrt{-1})^{\mu/\nu}$, where μ and ν are real integers. Nowhere in this article does he proceed to complex exponents.

Euler treats with remarkable care and clearness the relations between la, $l(-a)$, $2la$, $2l(-a)$, and la^2. He lets p be all even numbers, q all odd numbers. Then

$$l(+a) = A \pm p\pi\sqrt{-1}, \quad l(-a) = A \pm q\pi\sqrt{-1};$$

hence

$$l(+a)^2 = 2l(+a) = 2A \pm 2p\pi\sqrt{-1}, \quad l(-a)^2 = 2l(+a) = 2A \pm 2q\pi\sqrt{-1}.$$

But we have the general formula $la^2 = 2A \pm p\pi\sqrt{-1}$; hence $2l(+a) = la^2$ and $2l(-a) = la^2$, *prenant le signe de = pour marquer, que les valeurs de* $2l(-a)$ *ou de* $2l(+a)$ *se rencontrent parmi les valeurs de* la^2. He adds, *on ne sauroit dire à la verité qu'il soit* $2l(-a) = 2l(+a)$. A superficial comment upon this would be that here things equal to the same thing are *not* equal to each other. The reason why $2l(-a)$ and $2l(+a)$ are not equal is, of course, that $2q$ is never the same number as $2p$, even though both belong to the more general class p. Euler points out another difficulty. If we take $l(-a)^2 = la^2$, in the sense $2A \pm 2q\pi\sqrt{-1} = 2A \pm p\pi\sqrt{-1}$, then we do have $2l(-a) = 2la$, even though, as seen above, la is not $l(-a)$. A superficial comment on this would be that here the axiom—equals divided by equals give equals—breaks down. Euler is dealing here with equations which in the nineteenth century came to be called *incomplete* equations, in which each value on one side of an equation is equal to some value on the other side, but not *vice versa.* Euler proceeds to show under what conditions $2la = 2l(-a)$ becomes a *complete* equation. Let p and p' be even numbers, whether equal or not; let q and q' be odd numbers, whether equal or not. Then take $2l(+a) = 2A \pm (p + p')\pi\sqrt{-1}$, $2l(-a) = 2A \pm (q + q')\pi\sqrt{-1}$; moreover, select the numbers so that $p + p' = q + q'$. Then a complete equation $2l(-a) = 2l(+a)$ is obtained. *Par conséquent dans ce sens on pourra soutenir qu'il est* $2l(-a) = 2l(+a)$, *sans qu'il soit* $l(-a) = l(+a)$. The concept of complete and incomplete equations here created by Euler has not before been attributed to him. Euler did not introduce for the complete equation a special name, nor a special notation. As regards his complete equation $2l(-a) = 2la$, it is readily seen that it involves the condition $p + p' = q + q'$ which is not practical in an algebra of general logarithms. Euler enters into similar discussions for $l(-a)^2$, la^3, la^4.

Euler's fourth problem is, given a logarithm, to find the anti-logarithm x, whether x be real or complex. If the given logarithm is $g\sqrt{-1}$, and g is a multiple of π, then x is 1 or -1. If g is not such a multiple, *pour le trouver on n'a qu'à prendre un arc de cercle* $= g$, *le rayon étant* $= 1$ *& ayant cherché son sinus & cosinus, le nombre cherché sera* $x = \cos g + \sqrt{-1} \sin g$. If the logarithm is

$f + g\sqrt{-1}$, then x is the product of two numbers, one having the logarithm f, the other the logarithm $g\sqrt{-1}$.

We come now to Euler's third paper, *Recherches sur les racines imaginaires des équations.*[1] We have never seen this paper mentioned in historical articles on logarithms. It aims to give two proofs of the theorem that every equation has a root. It was discussed by C. F. Gauss in his memorable inaugural dissertation of 1799.[2] In the second proof Euler shows that expressions involving even complicated arithmetic operations and root extractions can always be brought to the form $m + n\sqrt{-1}$. This part contains new developments on logarithms, but Gauss, in his criticisms of Euler, has no occasion to take up logarithmic theory. Cantor, in his great history,[3] avowedly follows Gauss, hence makes no reference to logarithms. Lampe appears completely to have overlooked this paper of Euler. Nor have we seen any reference to it in the succeeding half century of discussion of logarithmic theory.

Euler generalizes his previous results by considering the general power, in which the base and exponent are both complex. He lets

$$(a + b\sqrt{-1})^{m+n\sqrt{-1}} = x + y\sqrt{-1},$$

where x and y are unknowns to be found. He takes the logarithm of both sides, then differentiates the members of the resulting equation, considering a, b, x and y as variables. Letting $\sqrt{aa + bb} = c$, $A \tan b/a = \phi$, $\sin\phi = b/c$, $\cos\phi = a/c$, he gets, upon equating the reals and the imaginaries,

$$x = c^m e^{-n\phi} \cos(m\phi + nlc),$$

$$y = c^m e^{-n\phi} \sin(m\phi + nlc).$$

Writing in place of ϕ the more general value $\phi + 2\lambda\pi$, he has *toutes les valeurs possibles* which his general power may take, *en donnant à* λ *successivement toutes les valeurs* 0, $\pm$ 1, $\pm$ 2, *etc., où il suffit de prendre pour* c^m *la seule valeur réelle et positive, qui y est renfermée.* We see here that Euler has no hesitation in taking the logarithm of the general power involving complex numbers and that he finds this general power to be infinitely many-valued—a remarkable result at that time.

Proceeding to the consideration of special cases under the general formula just derived, he obtains the result $(\sqrt{-1})^{\sqrt{-1}} = e^{-2\lambda\pi - \pi/2} = 0{,}2078795763507$ for $\lambda = 0$, *qui est d'autant plus remarquable, qu'elle est réelle, et qu'elle renferme même une infinité de valeurs réelles différentes.*

Thereupon, independently of any of his previous results, he devotes half a page to the derivation of the logarithms of a complex number. He assumes $l(a + b\sqrt{-1}) = x + y\sqrt{-1}$, x and y being unknown, then differentiates, and equates the reals and the imaginaries. Thus x and y are found, expressed in terms of arc sine and arc cosine. Here the great question which was

[1] *Histoire de l'acad. r. d. scien. et bell. lett.*, année 1749, à Berlin, 1751, pp. 222–288.
[2] See *Ostwald's Klassiker*, No. 14, Leipzig, 1890; Gauss, *Werke*, Vol. III, 1876, pp. 1–30.
[3] M. Cantor, *op. cit.*, Vol. III, 2. aufl., 1901, pp. 601–604.

agitated for a century was satisfactorily treated within the compass of less than a couple of dozen lines.

Did Euler communicate his results on logarithms to his old and revered master, John Bernoulli I? Historians have given no information on this question. But there is evidence that Euler did so. Pessuti, who took part in the controversy in Italy, states that he met Euler in St. Petersburg and that Euler made to him the statement that "having communicated his [Euler's] results to John Bernoulli shortly before his death, the good old man replied that he died content, because he saw reconciled what seemed to be irreconcilable paradoxes, namely the contradictions which had given rise to the great dispute on logarithms of negative numbers, long continued, between Leibniz and himself."[1] Calandrelli replies to Pessuti that this statement cannot be true, since Bernoulli died in 1748, and Euler's article was sent to the Berlin Academy in 1749. But we know now that Euler had his first paper completed in 1747.

(To be continued.)

AMICABLE NUMBER TRIPLES.[2]

By L. E. DICKSON, University of Chicago.

1. Two numbers are called amicable if each is the sum of the proper divisors of the other, a proper divisor of n being a divisor less than n. We shall say that three numbers form an amicable triple if the sum of the proper divisors of each equals the sum of the remaining two numbers. Let $\sigma(n)$ denote the sum of all the divisors of n. Then n_1, n_2, n_3 form an amicable triple if

$$\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = n_1 + n_2 + n_3. \tag{1}$$

Similarly, $n_1, \cdots, n_k$ form an amicable k-triple if

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k. \tag{2}$$

If $k = 2$, n_1 and n_2 are amicable numbers in the usual sense.

For $k > 2$, I have verified that the only solutions of (2) in which not every n_i exceeds 1,000 are those with $k = 3$, $n_1 = n_2 = n_3 = 120$ or 672. When the k numbers n_i are all equal, n_1 is called a multiply perfect number of multiplicity k; as many as 251 such numbers are known,[3] all with $k \leqq 7$.

I obtain below 8 sets of amicable triples in which two of the numbers are equal, and the triple of distinct numbers (end of § 8):

$$293 \cdot 337a, \quad 5 \cdot 16561a, \quad 99371a, \quad a = 2^5 \cdot 3 \cdot 13. \tag{I}$$

Another amicable triple of distinct numbers is obtained in § 14 by a device.

[1] G. Calandrelli, *Saggio analitico, etc.*, Roma, 1778, p. 14.

[2] Read before the American Mathematical Society, December 31, 1912.

[3] Cf. Carmichael and Mason, *Proceedings Indiana Academy of Science*, 1911, pp. 257–270.